Exploring Rational Functions

Part I - The numerator is a constant and the denominator is a linear factor.

1. The parent function for rational functions is the reciprocal function:

\[ y = \frac{1}{x} \]

Graph and analyze the reciprocal function:

Domain: __________________________

Range: __________________________

Maximums: ________________________

Minimums: ________________________

Bounded: _________________________

Horizontal Asymptotes: ______________

Vertical Asymptotes: ________________

End Behavior: \( \lim_{{x \to -\infty}} \frac{1}{x} = \) ____________ \( \lim_{{x \to \infty}} \frac{1}{x} = \) ____________

2. Graph the following transformed functions:

\[ y = \frac{1}{x-2} \quad y = \frac{1}{x+2} \quad y = \frac{1}{x-4} \quad y = \frac{1}{x+5} \quad y = \frac{1}{x-6} \quad y = \frac{1}{x+7} \]

Generalize the patterns you see by answering: How is \( y = \frac{1}{x-h} \) transformed from the parent function? Discuss the transformations and any changes to the analysis of the graph (in particular, asymptotes).

3. Graph the following transformed functions:

\[ y = \frac{4}{x} = 4\left(\frac{1}{x}\right) \quad y = \frac{10}{x} = 10\left(\frac{1}{x}\right) \quad y = \frac{1}{2x} = \frac{1}{2}\left(\frac{1}{x}\right) \quad y = \frac{-1}{x} = -1\left(\frac{1}{x}\right) \]

\[ y = -\frac{4}{x} = -4\left(\frac{1}{x}\right) \quad y = \frac{1}{-2x} = \left(-\frac{1}{2}\right)\left(\frac{1}{x}\right) \]

Generalize the patterns you see by answering: How is \( y = \frac{a}{x} \) transformed from the parent function? Discuss the transformations and any changes to the analysis of the graph.

4. Graph the following transformations:

\[ y = \frac{1}{x} + 3 \quad y = \frac{4}{x} - 5 \quad y = \frac{4}{x} + 2 \quad y = \frac{4}{x} - 6 \]

Generalize the patterns you see by answering: How is \( y = \frac{1}{x} + k \) transformed from the parent function? Discuss the transformations and any changes to the analysis of the graph (in particular, asymptotes).
Part II – Higher order polynomials in the denominator

1. Why are the two functions below the same? Reason using algebra!

\[ y = \frac{1}{(x+6)(x-4)} \quad y = \frac{1}{x^2+2x-24} \]

Practice: Write each function with a factored denominator.

\[ y = \frac{1}{x^2-2x-24} \quad y = \frac{1}{x^2+5x+6} \quad y = \frac{1}{x^2+7x+6} \quad y = \frac{1}{x^2-16} \]

2. Now graph all five functions:

\[ y = \frac{1}{x^2+2x-24} \quad y = \frac{1}{x^2-2x-24} \quad y = \frac{1}{x^2+5x+6} \quad y = \frac{1}{x^2+7x+6} \quad y = \frac{1}{x^2-16} \]

How do the vertical asymptotes (plural!) relate to the functions’ factored denominators?

Generalize your results by completing the sentence: Given a rational function in the form

\[ \frac{1}{(x-a)(x-b)\cdots(x-z)} \]

the function’s vertical asymptotes will be ____________________.

Part III – Zeros of rational functions

1. Recall how fractions “work”: Evaluate the following – without a calculator:

\[ y = \frac{0}{7} \quad y = \frac{2(4)-12}{4-7} \quad y = \frac{2-2}{2-7} \quad y = \frac{(4-10)(4-4)}{4-7} \]

Why did you get the same result when you evaluated each fraction?

2. For each function below, find the x-intercepts (zeros) of the function. (Hint: What is the only part of the rational function that needs to be zero?) You may use algebra or a graphing calculator. Then generalize the patterns you see by completing the sentence.

\[ y = \frac{x}{x-2} \quad y = \frac{x+3}{x-2} \quad y = \frac{x-4}{x-2} \quad y = \frac{x^2+5x+6}{x-2} \quad y = \frac{x^2-9}{x-2} \quad y = \frac{(x-3)(x-5)(x+1)}{x-2} \]

Given a rational function in the form: \( r(x) = \frac{f(x)}{g(x)} \), then the _______ of __________ are the __________ of __________ are the zeros of the rational function, \( r(x) \), as well. Why does this result make sense?
Part IV - Exploring horizontal and slant asymptotes
(which describe the END BEHAVIOR of the functions)

The degree of the numerator, in our text, is referred to as \( n \) and the degree of the denominator as \( m \). Examples:

\[
y = \frac{x^2 + 5x + 6}{x - 2} \quad n = 2 \text{ and } m = 1
\]

\[
y = \frac{(x-4)(x-3)(x+7)}{x^2 - 5} \quad n = 3 \text{ and } m = 2
\]

1. For each function below, state \( m \) and \( n \). Then graph and state the horizontal asymptote.

<table>
<thead>
<tr>
<th>Function</th>
<th>( n )</th>
<th>( m )</th>
<th>equation of the horizontal asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \frac{5(x-2)}{x^3 + 2x - 4} )</td>
<td>( 2 )</td>
<td>( 4 )</td>
<td>( y = \frac{1}{2} )</td>
</tr>
<tr>
<td>( y = \frac{4x^2}{x^5 - 3} )</td>
<td>( 2 )</td>
<td>( 5 )</td>
<td>( y = 0 )</td>
</tr>
<tr>
<td>( y = \frac{5x^2 - 7}{x^3 + 3} )</td>
<td>( 2 )</td>
<td>( 3 )</td>
<td>( y = \frac{5}{3} )</td>
</tr>
<tr>
<td>( y = \frac{8x^6}{x^7 + 2} )</td>
<td>( 6 )</td>
<td>( 7 )</td>
<td>( y = 0 )</td>
</tr>
<tr>
<td>( y = \frac{[5(x-3)(7x-4)]}{(x-2)(x+1)(x+5)(x-8)} )</td>
<td>( 5 )</td>
<td>( 8 )</td>
<td>( y = 0 )</td>
</tr>
</tbody>
</table>

In each case above, \( n \) \underline{\text{___________}} \( m \) and the horizontal asymptote is \( y = \underline{\text{_______}} \).

Choose \(<\), \(>\), or \(=\).

Generalize by completing the sentence: If \( n \underline{\text{______}} m \), then the horizontal asymptote will be \( y = \underline{\text{______}} \).

KEEP GOING!!!
2. Complete the table below. Graph the functions as needed to find the asymptotes.

<table>
<thead>
<tr>
<th>Function</th>
<th>n</th>
<th>m</th>
<th>Leading coefficient of the numerator</th>
<th>Leading coefficient of the denominator</th>
<th>Equation of the horizontal asymptote</th>
</tr>
</thead>
</table>
| \[
\frac{5x^2-7}{x^2+3}
\] |     |     |                                      |                                        |                                        |
| \[
\frac{10x^2-7}{5x^2+3}
\] |     |     |                                      |                                        |                                        |
| \[
\frac{18x^2}{6x^3+4}
\] |     |     |                                      |                                        |                                        |
| \[
\frac{3x^4+8x+9}{2x^4-1}
\] |     |     |                                      |                                        |                                        |
| \[
\frac{8x^4-2x^2+1}{2x^4+3}
\] |     |     |                                      |                                        |                                        |
| \[
\frac{8x^2-2x+1}{4x^2-3}
\] |     |     |                                      |                                        |                                        |

Generalize your results by completing the sentence: If \( n \) _______ \( m \), then the horizontal asymptote is \( y = \) _______. (You may use words to describe the result – or read section 2.6 for notation ideas.)

4. Slant asymptotes. Sometimes a rational function’s end behavior is not defined by horizontal asymptotes but by a slant asymptote.

All these functions have slant asymptotes. (Graph as many as you need to see what a slant asymptote looks like.) What is the relationship between \( n \) and \( m \)?

\[
\frac{5x^4-7}{x^3+3} \quad \frac{3x^2}{x+5} \quad \frac{5x^2-7}{x-1} \quad \frac{2x^3+3x-1}{5x^2} \quad \frac{-7x^6+5x}{2x^5+2}
\]

What needs to be true about the relationship between \( n \) and \( m \) for a rational function to have a slant asymptote?
1. Use long division to find the quotient of each rational function below. Do not worry about the remainder. Then graph the rational function in \( y_1 \) and its quotient in \( y_2 \) using the stated window. What do you notice about the end behavior of the rational function and its quotient?

\[
y = \frac{5x^4 - 7}{x^2 + 3} \quad x_{\text{min}} = -20, \ x_{\text{max}} = 10, \ y_{\text{min}} = -30, \ y_{\text{max}} = 30
\]

\[
y = \frac{3x^2}{x+3} \quad x_{\text{min}} = -30, \ x_{\text{max}} = 30, \ y_{\text{min}} = -200, \ y_{\text{max}} = 200
\]

\[
y = \frac{-7x^6 + 5x}{2x^6 + 2} \quad x_{\text{min}} = -10, \ x_{\text{max}} = 10, \ y_{\text{min}} = -10, \ y_{\text{max}} = 10
\]

2. Now try the same process with functions that do not have slant asymptotes for their end behavior.

\[
y = \frac{8x^6 + 4x^5 + 9}{2x^4 - 1} \quad x_{\text{min}} = -5, \ x_{\text{max}} = 5, \ y_{\text{min}} = -20, \ y_{\text{max}} = 20
\]

\[
y = \frac{6x^6 + 3x + 7}{3x^2 + 3} \quad x_{\text{min}} = -5, \ x_{\text{max}} = 5, \ y_{\text{min}} = -15, \ y_{\text{max}} = 15
\]

3. Generalize: Given a rational function, how can you find the End Behavior Function?

4. Why do you think the remainder is unimportant when describing the End Behavior of a Rational Function?

5. When you graphed the functions in Part II, their graphs had more than one vertical asymptote. When you graphed some of the functions in Part IV Question 2, their graphs did not have vertical asymptotes. Instead the graphs were continuous with local minimums or maximums. Based on these explorations, how can you determine by looking at the equation how many vertical asymptotes a rational function will have?

6. Expanding this idea:

Factor the numerator and denominator of the two functions below. What do you notice is a difference between the first function and the second function? Can you simplify the first function after you factor?

\[
y = \frac{x^2 + 2x - 15}{x^2 + 13x + 40} \quad y = \frac{x^2 + 2x - 15}{x^2 + 10x + 16}
\]

Graph each of the functions. Which function has two vertical asymptotes? Which has only one? Why do you think this happens?

Graph the function with one vertical asymptote. Zoom in as needed. What do you notice happening at the x-value where you would have expected the second vertical asymptote? (Depending on your calculator model, you may need to use dot mode, or you may not see the event.)

Go to the table. What do you notice happens at this x-value?

Why do you think this interesting result happens but the function does not have a vertical asymptote at this x-value? How would you indicate this result if you were graphing by hand?

7. Use the exploration in #6 to provide a more detailed answer to #5.
Rational Functions Worksheet:

**Due:**

Based on the explorations, provide the partial analysis required for the following functions. Do so **without** a calculator. Then, verify your results **with** a calculator.

<table>
<thead>
<tr>
<th>Function →</th>
<th>( \frac{3}{x+4} )</th>
<th>( \frac{x+7}{(x-4)(x+8)} )</th>
<th>( \frac{3x^2 + 2x + 1}{4x^2 + 20x + 24} )</th>
<th>( \frac{x^3 + x^2 + 4x + 4}{2x^2 - 8} )</th>
<th>( \frac{6(x-4)(x+3)(x-8)(x+10)}{5x^4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Zeros</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x-intercepts)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Horizontal Asymptotes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vertical Asymptotes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type of end behavior: horizontal or slant asymptote</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

2. State the degree of the numerator and the degree of the denominator for each function. Then, state the horizontal asymptote **OR** why the function does not have a horizontal asymptote.

<table>
<thead>
<tr>
<th>Function</th>
<th>m</th>
<th>n</th>
<th>Equation of horizontal asymptote OR Statement of why one does not exist</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \frac{3}{5x+7} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{7x^{100} + 5x^{5} - 8x + 9}{2x^{100} + 13x^{39}} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{-11x^5 + 9x^5 - 18x}{4x^{12} - 8x^5 + 1} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{31x^4 + 9x - 10}{2x + 12} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{x^4 + x^3 + x^2 + x + 1}{x^4 - 3x^2} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3. Match each graph on the next page with the correct equation below. Justify your answers by discussing vertical asymptotes, horizontal asymptotes, zeros, and end behavior. Yes! You do see more equations than graphs!

Equations:

<table>
<thead>
<tr>
<th>Equation 1</th>
<th>Equation 2</th>
<th>Equation 3</th>
<th>Equation 4</th>
<th>Equation 5</th>
<th>Equation 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \frac{-8x^4 + 2x^3 - 5x^2 + x}{6x^4 - 9x + 7}$</td>
<td>$y = -18 \frac{x^5 + 3}{6x^3 + 7}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$y = \frac{x+8}{x-2}$  

$y = \frac{x-8}{x+2}$  

$y = \frac{x+8}{x+2}$  

$y = \frac{x-8}{x-2}$  

$y = \frac{(x+8)(x+3)}{x-2}$  

$y = \frac{x+8}{(x-2)(x+3)}$

$y = \frac{5x}{3x-15}$  

$y = \frac{3x}{5x-15}$  

$y = \frac{8}{x-2}$  

$y = \frac{8}{x+2}$  

$y = \frac{x-8}{(x+2)(x+3)}$  

$y = \frac{x+7}{(x-2)(x-3)}$